RELATIONS FOR VIRTUAL FUNDAMENTAL CLASSES OF HILBERT SCHEMES OF CURVES ON SURFACES

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ABSTRACT. In [DKO] we constructed virtual fundamental classes $[[Hilb_V^m]]$ for Hilbert schemes of divisors of topological type m on a surface V, and used these classes to define the Poincaré invariant of V:

$$(P_V^+, P_V^-) : H^2(V, \mathbb{Z}) \longrightarrow \Lambda^* H^1(V, \mathbb{Z}) \times \Lambda^* H^1(V, \mathbb{Z})$$

We conjecture that this invariant coincides with the full Seiberg-Witten invariant computed with respect to the canonical orientation data.

In this note we prove that the existence of an integral curve $C \subset V$ induces relations between some of these virtual fundamental classes $[[Hilb_V^m]]$. The corresponding relations for the Poincaré invariant can be considered as algebraic analoga of the fundamental relations obtained in [OS].

1. Introduction

The symplectic Thom conjecture for homology classes with negative self-intersection, proven by Ozsváth and Szabó, is an immediate consequence of the following two facts:

i) Taubes' constraints for the Seiberg-Witten basic classes of a closed symplectic four-manifold [T].
ii) A fundamental relation between certain Seiberg-Witten invariants, which arises from embedded surfaces with negative self-intersection, due to Ozsváth and Szabó [OS].

In this note we prove an analoguous relation for the virtual fundamental classes of certain Hilbert schemes of algebraic curves on smooth projective surfaces. To be more precise: Let V be a smooth connected projective surface over $\mathbb{C}$. For any class $m \in H^2(V, \mathbb{Z})$ we have the Hilbert scheme Hilb_V^m parametrizing effective divisors $D \subset V$ with $c_1(\mathcal{O}_V(D)) = m$. In [DKO] we constructed a virtual fundamental class $[[\mathrm{Hilb}_V^m]] \in A_*(\mathrm{Hilb}_V^m)$ in the Chow group of Hilb_V^m. Note that there exists a natural morphism $\rho : \mathrm{Hilb}_V^m \to \mathrm{Pic}_V^m$ sending a divisor $D \subset V$ to the class $[\mathcal{O}_V(D)]$ of its associated line bundle. Let $\mathbb{D} \subset \mathrm{Hilb}_V^m \times V$ be the universal divisor, and put $u := c_1(\mathcal{O}_V(\mathbb{D})|_{\mathrm{Hilb}_V^m \times \{p\}})$, where $p \in V$ is an arbitrary point.

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Consider now an integral curve $C \subset V$, set $c := c_1(\mathcal{O}_V(C))$, and denote by $\kappa_c \in \Lambda^2 H^1(V,\mathbb{Z})^\vee$ the map:

$$\begin{array}{rcl} \kappa_c : \Lambda^2 H^1(V,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ a \wedge b & \longmapsto & \langle a \cup b \cup c, [V] \rangle . \end{array}$$

Let $\iota : \mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ be the closed embedding sending $D' \in \mathrm{Hilb}_V^{m-c}$ to $D' + C \in \mathrm{Hilb}_V^m$. Our main result relates $[[\mathrm{Hilb}_V^m]]$ and $[[\mathrm{Hilb}_V^{m-c}]]$ when $m \cdot c < 0$, and $[[\mathrm{Hilb}_V^m]]$ and $[[\mathrm{Hilb}_V^{m+c}]]$ when $(k-m)\cdot c < 0$. Here $k := c_1(\mathcal{K}_V)$ is the first Chern class of the canonical line bundle.

Theorem 3. *Let V be a surface, and fix a class $m \in H^2(V,\mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *Suppose that $m \cdot c < 0$, and denote by ρ the map $\mathrm{Hilb}_V^m \to \mathrm{Pic}_V^m$. Let $\iota : \mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$[[\mathrm{Hilb}_V^m]] = \left(\sum_i \rho^* \left(\frac{\kappa_c^i}{i!} \right) \cdot u^{\frac{c^2 + c\cdot m}{2} - m\cdot c - i} \right) \cap \iota_* [[\mathrm{Hilb}^{m-c}]].$$

ii) *Suppose that $(k-m)\cdot c < 0$, and denote by $\tilde{\rho}$ the map $\mathrm{Hilb}_V^{m+c} \to \mathrm{Pic}_V^{m+c}$. Let $\iota : \mathrm{Hilb}_V^m \to \mathrm{Hilb}_V^{m+c}$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$\iota_* [[\mathrm{Hilb}_V^m]] = \left(\sum_i \tilde{\rho}^* \left(\frac{(-\kappa_c)^i}{i!} \right) \cdot u^{\frac{c^2 + c\cdot k}{2} - (k-m)c - i} \right) \cap [[\mathrm{Hilb}_V^{m+c}]].$$

In [DKO] we used the virtual fundamental classes $[[\mathrm{Hilb}_V^m]]$ to define a map

$$(P_V^+, P_V^-) : H^2(V,\mathbb{Z}) \longrightarrow \Lambda^* H^1(V,\mathbb{Z}) \times \Lambda^* H^1(V,\mathbb{Z})$$

which we call the Poincaré invariant of V. This map is invariant under smooth deformations of V, satisfies a blow-up formula, and a wall crossing formula for surfaces with $p_g(V) = 0$. We conjecture that the Poincaré invariant coincides with the full Seiberg-Witten invariant of [OT] computed with respect to the canonical orientation data. Our relations between the virtual fundamental classes of Hilbert schemes lead to corresponding relations for the Poincaré invariant:

Theorem 6. *Let V be a surface, and fix a class $m \in H^2(V,\mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *If $m \cdot c < 0$, then*

$$P_V^{\pm}(m) = \tau_{m(m-k)} \left(\exp(\kappa_c) \cap P_V^{\pm}(m-c) \right).$$

ii) *If $(k-m)\cdot c < 0$, then*

$$P_V^{\pm}(m) = \tau_{m(m-k)} \left(\exp(-\kappa_c) \cap P_V^{\pm}(m+c) \right).$$

This result can be considered as an algebraic analog of the Ozsváth-Szabó relation, as we will explain in the section 4 below.

2. Comparing virtual fundamental classes of Hilbert schemes

In this paper all surfaces will be smooth, projective, connected, and defined over the field of complex numbers. We denote by $k := c_1(\mathcal{K}_V)$ the first Chern class of the canonical line bundle of a surface V.

Recall that an element $c \in H^2(V,\mathbb{Z})$ is *characteristic* iff $c \equiv k \mod 2$. For a characteristic element $c \in H^2(V,\mathbb{Z})$, we denote by $\theta_c \in \Lambda^2 H^1(V,\mathbb{Z})^\vee$ the map

$$\begin{array}{rcl} \theta_c : \Lambda^2 H^1(V,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ a \wedge b & \longmapsto & \frac{1}{2}\langle a \cup b \cup c, [V]\rangle. \end{array}$$

We define $\xi_V \in \Lambda^4 H^1(V,\mathbb{Z})^\vee$ to be the map

$$\begin{array}{rcl} \xi_V : \Lambda^4 H^1(V,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ a \wedge b \wedge c \wedge d & \longmapsto & \langle a \cup b \cup c \cup d, [V]\rangle. \end{array}$$

Lemma 1. *Let V be a surface, and fix a class $m \in NS(V)$. Choose a normalized Poincaré line bundle $\mathbb{L}$ on $\mathrm{Pic}^m_V \times V$, and let $\mu : \mathrm{Pic}^m_V \times V \to \mathrm{Pic}^m_V$ be the projection. Then we have*

$$ch(\mu_!\mathbb{L}) = \chi(\mathcal{O}_V) + \frac{m(m-k)}{2} - \theta_{2m-k} + \xi_V.$$

Proof. By the Grothendieck-Riemann-Roch theorem [F, Thm.15.2] we have

$$td(\mathrm{Pic}^m_V) \cdot ch(\mu_!\mathbb{L}) = \mu_! \left\{ td(\mathrm{Pic}^m_V \times V) \cdot ch(\mathbb{L}) \right\}.$$

Hence we need to compute those components of the expression

$$td(\mathrm{Pic}^m_V \times V) \cdot ch(\mathbb{L})$$

which have bidegree $(*, 4)$ with respect to the decomposition

$$\begin{array}{rcl} H^*(\mathrm{Pic}^m_V \times V, \mathbb{Z}) & \cong & H^*(\mathrm{Pic}^m_V, \mathbb{Z}) \otimes H^*(V,\mathbb{Z}) \\ & \cong & \Lambda^* H^1(V,\mathbb{Z})^\vee \otimes H^*(V,\mathbb{Z}). \end{array}$$

Set $f := c_1(\mathbb{L})$. Then

$$\begin{array}{rcl} f^{2,0} & = & 0 \in H^2(\mathrm{Pic}^m_V, \mathbb{Z}), \\ f^{1,1} & = & id \in \mathrm{Hom}(H^1(V,\mathbb{Z}), H^1(V,\mathbb{Z})), \\ f^{0,2} & = & m \in H^2(V,\mathbb{Z}), \end{array}$$

where the first equality holds since $\mathbb{L}$ is normalized.

Next we compute $g := f^2$. We obtain

$$\begin{array}{rcl} g^{2,2} & = & -2 \cdot (a \wedge b \mapsto a \cup b) \in \mathrm{Hom}(\Lambda^2 H^1(V,\mathbb{Z}), H^2(V,\mathbb{Z})), \\ g^{1,3} & = & 2 \cdot (a \mapsto a \cup m) \in \mathrm{Hom}(H^1(V,\mathbb{Z}), H^3(V,\mathbb{Z})), \\ g^{0,4} & = & m \cup m \in H^4(V,\mathbb{Z}), \end{array}$$

all other components being zero. Here the first equality needs justification. Choose a basis $v_1, \ldots, v_{2q}$ of $H^1(V,\mathbb{Z})$, and denote by $w_1, \ldots, w_{2q}$ the dual basis of $H^1(V,\mathbb{Z})^\vee$. Then

$$f^{1,1} = \sum_i w_i \otimes v_i,$$

and

$$\begin{aligned} g^{2,2} &= \left(f^{1,1}\right)^2 \\ &= (\sum_i w_i \otimes v_i) \cup (\sum_i w_i \otimes v_i) \\ &= -\sum_i \sum_j (w_i \wedge w_j) \otimes (v_i \cup v_j) \\ &= -2 \sum_{i<j} (w_i \wedge w_j) \otimes (v_i \cup v_j). \end{aligned}$$

Now we compute the component of f^3 of bidegree $(2,4)$, the only component that does not vanish. We find

$$\begin{aligned} f^3 &= 3(f^{1,1})^2 \cup f^{0,2} \\ &= -6 \cdot (a \wedge b \mapsto a \cup b \cup m) \in \mathrm{Hom}(\Lambda^2 H^1(V,\mathbb{Z}), H^4(V,\mathbb{Z})). \end{aligned}$$

Finally we obtain

$$\begin{aligned} f^4 &= (f^{1,1})^4 \\ &= \sum_{i,j,k,l} (w_i \wedge w_j \wedge w_k \wedge w_l) \otimes (v_i \cup v_j \cup v_k \cup v_l) \\ &= 24 \left(\sum_{i<j<k<l} (w_i \wedge w_j \wedge w_k \wedge w_l) \otimes (v_i \cup v_j \cup v_k \cup v_l) \right) \\ &= 24(a \wedge b \wedge c \wedge d \mapsto a \cup b \cup c \cup d). \end{aligned}$$

Since $td(\mathrm{Pic}^m_V) = 1$, we get

$$\begin{aligned} td(\mathrm{Pic}^m_V \times V) &= pr_V^* td(V) \\ &= pr_V^*(1 - \frac{1}{2}k + \chi(\mathcal{O}_V) \cdot PD[pt]), \end{aligned}$$

where $pr_V : \mathrm{Pic}^m_V \times V \to V$ denotes the projection onto V.

Putting everything together, we get

$$\begin{aligned} ch(\mu_!\mathbb{L}) &= \left\{ \exp f \cup pr_V^* \left(1 - \frac{k}{2} + \chi(\mathcal{O}_V) \cdot PD[pt] \right) \right\} / [V] \\ &= \left\{ (\exp f)^{*,4} - (\exp f)^{*,2} \cup pr_V^* \frac{k}{2} + \chi(\mathcal{O}_V) \cdot PD[pt] \right\} / [V] \\ &= \chi(\mathcal{O}_V) + \frac{m \cdot (m-k)}{2} - \theta_{2m-k} + \xi_V. \end{aligned}$$

□

For an arbitrary element $c \in H^2(V,\mathbb{Z})$, we denote by $\kappa_c \in \Lambda^2 H^1(V,\mathbb{Z})^\vee$ the map

$$\begin{aligned} \kappa_c : \Lambda^2 H^1(V,\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ a \wedge b &\longmapsto \langle a \cup b \cup c, [V] \rangle. \end{aligned}$$

Corollary 2. *Let V be a surface, and fix two classes $m, c \in NS(V)$. Choose a normalized Poincaré line bundle $\mathbb{L}$ on $\mathrm{Pic}_V^m \times V$ and a line bundle $\mathcal{L}_c$ on V with $c_1(\mathcal{L}_c) = c$. Let $\mu : \mathrm{Pic}_V^m \times V \to \mathrm{Pic}_V^m$ and $pr_V : \mathrm{Pic}_V^m \times V \to V$ be the projections. Then*

$$\begin{aligned} ch(\mu_!\mathbb{L} - \mu_!(\mathbb{L} \otimes pr_V^*\mathcal{L}_c^\vee)) &= m \cdot c - \frac{c^2 + c \cdot k}{2} - \kappa_c, \\ c(\mu_!\mathbb{L} - \mu_!(\mathbb{L} \otimes pr_V^*\mathcal{L}_c^\vee)) &= \exp(-\kappa_c). \end{aligned}$$

Proof. The assertion concerning the Chern character is a direct consequence of Lemma 1. The formula for the Chern class follows immediately since $H^*(\mathrm{Pic}_V^m, \mathbb{Z})$ has no torsion. □

In order to state our main result, we have to recall some facts from [DKO].

For a surface V and a class $m \in H^2(V,\mathbb{Z})$, we denote by Hilb_V^m the Hilbert scheme of divisors D with $c_1(\mathcal{O}_V(D)) = m$. Let $\mathbb{D} \subset \mathrm{Hilb}_V^m \times V$ be the universal divisor, and denote by $\pi : \mathrm{Hilb}_V^m \times V \to \mathrm{Hilb}_V^m$ the projection onto Hilb_V^m.

In [DKO], we constructed an obstruction theory (in the sense of Behrend and Fantechi)

$$\varphi : (R^\bullet \pi_* \mathcal{O}_{\mathbb{D}}(\mathbb{D}))^\vee \to \mathcal{L}^\bullet_{\mathrm{Hilb}_V^m}$$

for Hilb_V^m, and showed that this obstruction theory defines a virtual fundamental class

$$[[\mathrm{Hilb}_V^m]] \in A_{\frac{m(m-k)}{2}}(\mathrm{Hilb}_V^m).$$

Choose a point $p \in V$ and set

$$u := c_1(\mathcal{O}(\mathbb{D})|_{\mathrm{Hilb}_V^m \times \{p\}}).$$

Theorem 3. *Let V be a surface, and fix a class $m \in H^2(V,\mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *Suppose that $m \cdot c < 0$, and denote by ρ the map $\mathrm{Hilb}_V^m \to \mathrm{Pic}_V^m$. Let $\iota : \mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$[[\mathrm{Hilb}_V^m]] = \left(\sum_i \rho^* \left(\frac{\kappa_c^i}{i!} \right) \cdot u^{\frac{c^2 + c \cdot m}{2} - m \cdot c - i} \right) \cap \iota_*[[\mathrm{Hilb}^{m-c}]].$$

ii) *Suppose that $(k - m) \cdot c < 0$, and denote by $\tilde{\rho}$ the map $\mathrm{Hilb}_V^{m+c} \to \mathrm{Pic}_V^{m+c}$. Let $\iota : \mathrm{Hilb}_V^m \to \mathrm{Hilb}_V^{m+c}$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$\iota_*[[\mathrm{Hilb}_V^m]] = \left(\sum_i \tilde{\rho}^* \left(\frac{(-\kappa_c)^i}{i!} \right) \cdot u^{\frac{c^2 + c \cdot k}{2} - (k-m)c - i} \right) \cap [[\mathrm{Hilb}_V^{m+c}]].$$

Proof. Suppose first that $m \cdot c < 0$. Then we have $H^0(\mathcal{O}_C(D)) = 0$ for any divisor $D \in \mathrm{Hilb}_V^m$. It follows that the inclusion $\mathrm{Hilb}_V^{m-c} \to Hilb_V^m$ is an isomorphism. However, the obstruction theories differ: Denote by $\mathbb{C}$ the product $\mathrm{Hilb}_V^m \times C$. The short exact sequence

$$0 \to \mathcal{O}_{\mathbb{D}-\mathbb{C}}(\mathbb{D}-\mathbb{C}) \to \mathcal{O}_{\mathbb{D}}(\mathbb{D}) \to \mathcal{O}_{\mathbb{C}}(\mathbb{D}) \to 0$$

gives rise to a distinguished triangle:

$$\begin{tikzcd} R^\bullet\pi_*\mathcal{O}_{\mathbb{D}-\mathbb{C}}(\mathbb{D}-\mathbb{C}) \arrow[r] & R^\bullet\pi_*\mathcal{O}_{\mathbb{D}}(\mathbb{D}) \arrow[d] \\ & R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D}) \arrow[ul, "{[1]}"] \end{tikzcd}$$

Here $\pi : \mathrm{Hilb}_V^m \times V \to \mathrm{Hilb}_V^m$ is the projection. By the excess intersection formula [DKO, Prop.1.16], we have

$$[[\mathrm{Hilb}_V^m]] = c_{top}(R^1\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D})) \cap \iota_*[[\mathrm{Hilb}_V^{m-c}]].$$

The complex $R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D}))$ is the mapping cone of the morphism

$$R^\bullet\pi_*\mathcal{O}(\mathbb{D}-\mathbb{C}) \to R^\bullet\pi_*\mathcal{O}(\mathbb{D}).$$

Fix a normalized Poincaré line bundle $\mathbb{L}$ on $\mathrm{Pic}_V^m \times V$. Using [DKO, Lemma 3.15], we see that this choice endows Hilb_V^m with a relatively ample sheaf $\mathcal{O}_{\mathbb{L}}(1)$. Furthermore, there exists an isomorphism

$$\mathcal{O}(\mathbb{D}) \xrightarrow{\cong} (\rho \times id_V)^*\mathbb{L} \otimes \pi^*\mathcal{O}_{\mathbb{L}}(1),$$

and, since $\mathbb{L}$ is normalized, we have

$$u = c_1(\mathcal{O}_{\mathbb{L}}(1)).$$

This implies that $R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D})$ is the mapping cone of

$$\rho^*(R^\bullet\mu_*(\mathbb{L} \otimes pr_V^*\mathcal{O}_V(-C))) \otimes \mathcal{O}_{\mathbb{L}}(1) \to \rho^*(R^\bullet\mu_*\mathbb{L}) \otimes \mathcal{O}_{\mathbb{L}}(1).$$

Using Cor. 2 we conclude

$$c_{top}(R^1\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D})) = \sum_i \rho^*\left(\frac{\kappa_c^i}{i!}\right) \cdot u^{\frac{c^2+c\cdot m}{2} - m\cdot c - i},$$

which proves part i).

Suppose now that $(k-m)\cdot c < 0$. Then we have $H^1(\mathcal{O}_C(D)) = 0$ for any divisor $D \in \mathrm{Hilb}_V^{m+c}$. Denote by $\tilde{\mathbb{D}} \subset \mathrm{Hilb}_V^{m+c} \times V$ the universal divisor, and let $\tilde{\pi} : \mathrm{Hilb}_V^{m+c} \times V \to \mathrm{Hilb}_V^{m+c}$ be the projection. It follows that the sheaf $R^1\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c} \times C}(\tilde{\mathbb{D}})$ vanishes, and that $\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c} \times C}(\tilde{\mathbb{D}})$ is locally free. Moreover, ι induces an isomorphism

$$\mathrm{Hilb}_V^m \xrightarrow{\cong} Z(\lambda),$$

where λ is the canonical section in $\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c} \times C}(\tilde{\mathbb{D}})$.

The short exact sequence

$$0 \to \mathcal{O}_{\mathbb{D}}(\mathbb{D}) \to \mathcal{O}_{\mathbb{D}+\mathbb{C}}(\mathbb{D}+\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}(\mathbb{D}+\mathbb{C}) \to 0$$

gives rise to the following distinguished triangle:

$$\begin{tikzcd} R^\bullet\pi_*\mathcal{O}_{\mathbb{D}}(\mathbb{D}) \arrow[r] & R^\bullet\pi_*\mathcal{O}_{\mathbb{D}+\mathbb{C}}(\mathbb{D}+\mathbb{C}) \arrow[d] \\ & R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D}+\mathbb{C}) \arrow[ul, "{[1]}"] \end{tikzcd}$$

Hence functoriality [KKP, Thm.1] yields

$$\iota_*[[Hilb_V^m]] = c_{top}(\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c}\times C}(\tilde{\mathbb{D}})) \cap [[\mathrm{Hilb}_V^{m+c}]].$$

Fix again a normalized Poincaré line bundle $\mathbb{L}$ on Pic_V^m. By arguments similar to those of the first part, we see that $R^\bullet\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c}\times C}(\tilde{\mathbb{D}})$ is the mapping cone of

$$\tilde{\rho}^*(R^\bullet\mu_*\mathbb{L})\otimes\mathcal{O}_{\mathbb{L}\otimes pr_V^*\mathcal{O}_V(C)}(1) \to \tilde{\rho}^*(R^\bullet\mu_*(\mathbb{L}\otimes pr_V^*\mathcal{O}_V(C))\otimes\mathcal{O}_{\mathbb{L}\otimes pr_V^*\mathcal{O}_V(C)}(1).$$

Now Cor. 2 implies

$$c_{top}(\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c}\times C}(\tilde{\mathbb{D}})) = \sum_i \tilde{\rho}^*\left(\frac{(-\kappa_c)^i}{i!}\right)\cdot u^{\frac{c^2+c\cdot k}{2}-(k-m)c-i}$$

□

Remark 4. When C is rational, i.e. when the normalization $\hat{C}$ is isomorphic to $\mathbb{P}^1$, then $\kappa_c = 0$. When C is isomorphic to $\mathbb{P}^1$ and $c^2 \in \{0,-1\}$, then $m\cdot c < 0$ or $(k-m)\cdot c < 0$ for any $m \in H^2(V,\mathbb{Z})$.

To see this, let $j : \hat{C} \to V$ be the map induced by the inclusion $C \subset V$. Then for all $a, b \in H^1(V,\mathbb{Z})$

$$\begin{aligned} \kappa_c(a\wedge b) &= \langle a\cup b, j_*[\hat{C}]\rangle \\ &= \langle j^*a\cup j^*b, [\hat{C}]\rangle. \end{aligned}$$

Since the curve $\hat{C}$ is simply connected, the pull-backs j^*a and j^*b vanish, and therefore

$$\kappa_c(a\wedge b) = 0.$$

When C is isomorphic to $\mathbb{P}^1$ and $c^2 \in \{0,-1\}$, the adjunction formula yields $k\cdot c < 0$. This proves the second claim.

3. Relations for Poincaré invariants and the adjunction inequality

First we recall the definition of the Poincaré invariant. Let V be a surface, $p \in V$ an arbitrary point. Fix a class $m \in H^2(V,\mathbb{Z})$, denote by $\mathbb{D}^+$ the universal divisor over the Hilbert scheme Hilb_V^m, and set

$$u^+ := c_1\left(\mathcal{O}(\mathbb{D}^+)|_{\mathrm{Hilb}_V^m\times\{p\}}\right) \in H^2(\mathrm{Hilb}_V^m,\mathbb{Z}).$$

Since V is connected, the class u^+ does not depend on the chosen point p. Likewise, denote by $\mathbb{D}^-$ the universal divisor over the Hilbert scheme Hilb_V^{k-m}, where $k = c_1(\mathcal{K}_V)$. Put

$$u^- := c_1\left(\mathcal{O}(\mathbb{D}^-)|_{\mathrm{Hilb}_V^{k-m}\times\{p\}}\right) \in H^2(\mathrm{Hilb}_V^{k-m},\mathbb{Z}).$$

Denote by $\rho^\pm$ the following morphisms:

$$\begin{array}{rcl} \rho^+ : \mathrm{Hilb}_V^m & \longrightarrow & \mathrm{Pic}_V^m \\ D & \longmapsto & [\mathcal{O}_V(D)] \end{array}$$

$$\begin{array}{rcl} \rho^- : \mathrm{Hilb}_V^{k-m} & \longrightarrow & \mathrm{Pic}_V^m \\ D' & \longmapsto & [\mathcal{K}_V(-D')] \end{array}$$

By abuse of notation, we will denote the image of $[[\mathrm{Hilb}_V^m]]$ under the cycle map $A_*(\mathrm{Hilb}_V^m) \to H_*(\mathrm{Hilb}_V^m,\mathbb{Z})$ by the same symbol.

Definition 5. Let V be a surface. The *Poincaré invariant* of V is the map

$$\begin{array}{rcl} (P_V^+,P_V^-) : H^2(V,\mathbb{Z}) & \longrightarrow & \Lambda^* H^1(V,\mathbb{Z}) \times \Lambda^* H^1(V,\mathbb{Z}) \\ m & \longmapsto & (P_V^+(m), P_V^-(m)), \end{array}$$

defined by

$$P_V^+(m) := \rho_*^+\left(\sum_i (u^+)^i \cap [[Hilb_V^m]]\right)$$

and

$$P_V^-(m) := (-1)^{\chi(\mathcal{O}_V)+\frac{m(m-k)}{2}}\rho_*^-\left(\sum_i (-u^-)^i \cap [[Hilb_V^{k-m}]]\right),$$

if $m \in NS(V)$, and by $P_V^\pm(m) := 0$ otherwise.

For an integer n we define a truncation map

$$\tau_{\leq n} : \Lambda^* H^1(V,\mathbb{Z}) \longrightarrow \Lambda^* H^1(V,\mathbb{Z})$$

as follows: when $P = \sum_i P_i$ is the decomposition of a form P into its homogeneous components $P_i \in \Lambda^i H^1(V,\mathbb{Z})$, then

$$\tau_{\leq n}(P) := \sum_{i=0}^n P_i.$$

Theorem 6. *Let V be a surface, and fix a class $m \in H^2(V,\mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *If $m \cdot c < 0$, then*

$$P_V^\pm(m) = \tau_{m(m-k)}\left(\exp(\kappa_c) \cap P_V^\pm(m-c)\right).$$

ii) *If $(k-m)\cdot c < 0$, then*

$$P_V^\pm(m) = \tau_{m(m-k)}\left(\exp(-\kappa_c) \cap P_V^\pm(m+c)\right).$$

Proof. Suppose that $m\cdot c<0$, and let ι^+ be the inclusion $\mathrm{Hilb}_V^{m-c}\to\mathrm{Hilb}_V^m$. By part $i)$ of Thm. 3 we have

$$\begin{aligned}
P_V^+(m) &= \rho^+_*\left(\sum_i u^i\cap[[\mathrm{Hilb}_V^m]]\right)\\
&= \rho^+_*\left(\sum_i u^i\cap\left(\sum_j(\rho^+)^*\left(\frac{\kappa_c^j}{j!}\right)u^{\frac{c^2+c\cdot k}{2}-m\cdot c-j}\right)\cap\iota^+_*[[\mathrm{Hilb}_V^{m-c}]]\right)\\
&= \sum_j\frac{\kappa_c^j}{j!}\cap\rho^+_*\left(\sum_i u^{i+\frac{c^2+c\cdot m}{2}-m\cdot c-j}\cap\iota^+_*[[\mathrm{Hilb}_V^{m-c}]]\right)\\
&= \tau_{m(m-k)}\left(\exp(\kappa_c)\cap P_V^+(m-c)\right).
\end{aligned}$$

Let ι^- be the inclusion $\mathrm{Hilb}_V^{k-m}\to\mathrm{Hilb}_V^{k-m+c}$, and set $\epsilon:=(-1)^{\chi(\mathcal{O}_V)+\frac{m(m-k)}{2}}$. Note that under the isomorphism

$$\begin{array}{rcl}
\mathrm{Pic}_V^m & \longrightarrow & \mathrm{Pic}_V^{k-m}\\
{[\mathcal{L}]} & \longmapsto & [\mathcal{K}_V\otimes\mathcal{L}^\vee]
\end{array}$$

the cohomology class κ_c is mapped to κ_c, since this class is of degree 2. Hence part $ii)$ of Thm. 3 yields

$$\begin{aligned}
P_V^-(m) &= \epsilon\cdot(\rho^-)_*\left(\sum_i(-u)^i\cap\iota^-_*[[\mathrm{Hilb}_V^{k-m}]]\right)\\
&= \epsilon\cdot\rho^-_*\left(\sum_i(-u)^i\cap\left(\sum_j(\rho^-)^*\left(\frac{(-\kappa_c)^j}{j!}\right)\cdot u^{\frac{c^2+c\cdot k}{2}-m\cdot c-j}\cap[[\mathrm{Hilb}_V^{k-m+c}]]\right)\right)\\
&= \epsilon\cdot(-1)^{\frac{c^2+c\cdot k}{2}-m\cdot c}\left(\sum_j\frac{\kappa_c^j}{j!}\cap\rho^-_*\left(\sum_i(-u)^{i+\frac{c^2+c\cdot k}{2}-m\cdot c-j}\cap[[\mathrm{Hilb}_V^{k-m+c}]]\right)\right)\\
&= \tau_{m(m-k)}(\exp(\kappa_c)\cap P_V^-(m-c)).
\end{aligned}$$

The proof in the case $(k-m)\cdot c<0$ is similar. We omit the details. □

Recall that a class $m\in H^2(V,\mathbb{Z})$ is basic for a surface V, if

$$(P_V^+(m),P_V^-(m))\neq(0,0).$$

The surface V is of simple type if all basic classes $m\in H^2(V,\mathbb{Z})$ satisfy $m(m-k)=0$. In [DKO, Prop.6.25] we have shown that surfaces with $p_g(V)>0$ are of simple type. The following result can be considered as an algebraic analog of the Ozsváth-Szabó inequality [OS, Cor.1.7].

Proposition 7. *Let V be a surface with $p_g(V)>0$, let $C\subset V$ be a curve, and set $c:=c_1(\mathcal{O}_V(C))$. For any basic class $m\in H^2(V,\mathbb{Z})$ we have*

$$0\leq m\cdot c\leq k\cdot c,$$

unless C is a smooth rational curve. In this case we have

$$-1 \leq m \cdot c \leq k \cdot c + 1$$

for all basic classes $m \in H^2(V, \mathbb{Z})$.

Proof. Assume first that m is a basic class with $m \cdot c < 0$. Then Thm. 6 implies that also $m - c$ is a basic class. We have

$$\frac{(m-c)(m-c-k)}{2} = \frac{m(m-k)}{2} + p_a(C) - 1 - m \cdot c$$

Since any surface V with $p_g(V) > 0$ is of simple type, this implies

$$p_a(C) = 0 \text{ and } m \cdot c = -1.$$

Analoguously, if m is a basic class with $m \cdot c > k \cdot c$, then also $m + c$ is a basic class. Because

$$\frac{(m+c)(m+c-k)}{2} = \frac{m(m-k)}{2} + p_a(C) - 1 - (k-m) \cdot c,$$

we obtain this time

$$p_a(C) = 0 \text{ and } (k-m) \cdot c = -1.$$

□

4. Connection with the Ozsváth-Szabó relation

In order to explain the connection between Thm. 6 and the Ozsváth-Szabó relation, we briefly recall the structure of the full Seiberg-Witten invariants; for the construction and details, we refer to [OT].

Let (M, g) be a closed oriented Riemannian 4-manifold with first Betti number b_1. We denote by b_+ the dimension of a maximal subspace of $H^2(M, \mathbb{R})$ on which the intersection form is positive definite. Recall that the set of isomorphism classes of $Spin^c(4)$-structures on (M, g) has the structure of a $H^2(M, \mathbb{Z})$-torsor. This torsor does, up to a canonical isomorphism, not depend on the choice of the metric g and will be denoted by $Spin^c(M)$.

We have the Chern class mapping

$$\begin{aligned} c_1 : Spin^c(M) &\longrightarrow H^2(M, \mathbb{Z}) \\ \mathfrak{c} &\longmapsto c_1(\mathfrak{c}), \end{aligned}$$

whose image consists of all characteristic elements.

If $b_+ > 1$, then the Seiberg-Witten invariants are maps

$$SW_{M,\mathcal{O}} : Spin^c(M) \longrightarrow \Lambda^* H^1(M, \mathbb{Z}),$$

where $\mathcal{O}$ is an orientation parameter.

When $b_+ = 1$, then the invariants depend on a chamber structure and are maps

$$(SW^+_{M,(\mathcal{O}_1,\mathbf{H}_0)}, SW^-_{M,(\mathcal{O}_1,\mathbf{H}_0)}) : Spin^c(M) \longrightarrow \Lambda^* H^1(M, \mathbb{Z}) \times \Lambda^* H^1(M, \mathbb{Z}),$$

where $(\mathcal{O}_1, \mathbf{H}_0)$ are again orientation data. The difference of the two components is a purely topological invariant.

Let $\Sigma \subset M$ be a smoothly embedded, oriented, closed two-manifold. Fix a standard symplectic basis for $H_1(\Sigma, \mathbb{Z})$ and let $\{A_i, B_i\}_{i=1}^g$ be its image in $H^1(M, \mathbb{Z})^\vee$. We define the class $\theta(\Sigma) \in \Lambda^2 H^1(M, \mathbb{Z})^\vee$ by

$$\theta(\Sigma) = \sum_i A_i \wedge B_i.$$

Theorem 8 (Ozsváth-Szabó)**.** *Let M be a closed, oriented, smooth four-manifold with $b_+ > 0$, and let $\Sigma \subset M$ be a smoothly embedded, oriented, closed two-manifold of genus $g > 0$ with negative self-intersection*

$$[\Sigma] \cdot [\Sigma] = -n.$$

If $b_+ > 1$, then for each $Spin^c(4)$-structure $\mathfrak{c}$ with expected dimension $d(\mathfrak{c}) \geq 0$ and

$$|\langle c_1(\mathfrak{c}), [\Sigma] \rangle| \geq 2g + n$$

we have

$$SW_{M,\mathcal{O}}(\mathfrak{c}) = \tau_{\leq d(\mathfrak{c})}(\exp(\theta(\epsilon\Sigma)) \cap SW_{M,\mathcal{O}}(\mathfrak{c} + \epsilon PD(\Sigma))),$$

where $\epsilon = \pm 1$ is the sign of $\langle c_1(\mathfrak{c}), [\Sigma] \rangle$, and $PD(\Sigma)$ denotes the class Poincaré dual to $[\Sigma]$.

If $b_+ = 1$, then for each $Spin^c(4)$-structure $\mathfrak{c}$ with expected dimension $d(\mathfrak{c}) \geq 0$ and

$$|\langle c_1(\mathfrak{c}), [\Sigma] \rangle| \geq 2g + n$$

we have

$$SW^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) = \tau_{\leq d(\mathfrak{c})}(\exp(\theta(\epsilon\Sigma)) \cap SW^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c} + \epsilon PD[\Sigma])).$$

We need the following

Lemma 9. *Let M be a closed, oriented, smooth four-manifold. Let $\Sigma \subset M$ be a smoothly embedded, oriented, closed two-manifold, and let c be the Poincaré dual of the homology class $[\Sigma]$. Then*

$$\theta(\Sigma)(a \wedge b) = \langle a \cup b \cup c, [M] \rangle \quad \forall a, b \in H^1(M, \mathbb{Z}).$$

Proof. Fix a standard symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^g$, and let $\{A_i, B_i\}_{i=1}^g$ be its image in $H^1(M, \mathbb{Z})^\vee$. Then for all $a, b \in H^1(M, \mathbb{Z})$

$$\begin{aligned}
\langle a \cup b \cup c, [M] \rangle &= \langle a \cup b, c \cap [M] \rangle \\
&= \langle a \cup b, j_*[\Sigma] \rangle \\
&= \langle j^*a \cup j^*b, [\Sigma] \rangle \\
&= \sum_{i=1}^g \det \begin{pmatrix} j^*a(\alpha_i) & j^*a(\beta_i) \\ j^*b(\alpha_i) & j^*b(\beta_i) \end{pmatrix} \\
&= \sum_{i=1}^g \det \begin{pmatrix} A_i(a) & B_i(a) \\ A_i(b) & B_i(b) \end{pmatrix} \\
&= \theta(\Sigma)(a \wedge b).
\end{aligned}$$

□

At this point it is clear, that Thm. 6 and Thm. 8 are fomally analoguous statements. We believe however, that the actual source of this analogy is the conjectured equivalence between our Poincaré invariants and the full Seiberg-Witten invariants. To be precise, let V be a surface. Any *Hermitian* metric g on V defines a *canonical* $Spin^c(4)$-structure on (V, g). Its class $\mathfrak{c}_{can} \in Spin^c(V)$ does not depend on the choice of the metric. The Chern class of $\mathfrak{c}_{can}$ is $c_1(\mathfrak{c}_{can}) = -c_1(\mathcal{K}_V) = -k$.

Since $Spin^c(V)$ is a $H^2(V, \mathbb{Z})$-torsor, the distinguished element $\mathfrak{c}_{can}$ defines a bijection:

$$\begin{array}{ccc} H^2(V, \mathbb{Z}) & \longrightarrow & Spin^c(V) \\ m & \longmapsto & \mathfrak{c}_m \end{array}$$

The Chern class of the twisted structure $\mathfrak{c}_m$ is $2m - k$. Recall that any surface defines canonical orientation data $\mathcal{O}$ and $(\mathcal{O}_1, \mathbf{H}_0)$ respectively.

The precise conjectured relation between Poincaré and Seiberg-Witten invariants is:

Conjecture 10. *Let V be a surface, and denote by $\mathcal{O}$ or $(\mathcal{O}_1, \mathbf{H}_0)$ the canonical orientation data. If $p_g(V) = 0$, then*

$$P_V^{\pm}(m) = SW^{\pm}_{V,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}_m) \quad \forall m \in H^2(V, \mathbb{Z}).$$

If $p_g(V) > 0$, then

$$P_V^{+}(m) = P_V^{-}(m) = SW_{V,\mathcal{O}}(\mathfrak{c}_m) \quad \forall m \in H^2(V, \mathbb{Z}).$$

If this conjecture holds, Thm. 6 is essentially a consequence of Thm. 8. To see this, let $C \subset V$ be an integral curve in the surface V. Its arithmetic genus is given by the adjunction formula

$$p_a(C) = \frac{c^2 + c \cdot k}{2} + 1,$$

where $c := c_1(\mathcal{O}_V(C))$. Hence the inequality

$$|\langle c_1(\mathfrak{c}), [\Sigma]\rangle| \geq 2g + n$$

with $n = -[\Sigma] \cdot [\Sigma]$ reads

$$|\langle c_1(\mathfrak{c}), [\Sigma]\rangle| \geq c \cdot k + 2.$$

When $\mathfrak{c} = \mathfrak{c}_m$ for some $m \in H^2(V, \mathbb{Z})$, this means

$$|(2m - k) \cdot c| \geq c \cdot k + 2,$$

or equivalently

$$m \cdot c \leq -1 \text{ or } (k - m) \cdot c \leq -1.$$

Moreover, in the first case $\epsilon = -1$, whereas in the second case $\epsilon = +1$.

Conversely, Thm. 6 yields further evidence for the truth of Conj. 10.

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